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# Disentangling the control electron in a two qubit solid state quantum gate

**Seb Savory**

Optical Networks Group, UCL Department of Electronic and Electrical Engineering, University College London, Torrington Place, London, WC1E 7JE, UK

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**Abstract**
We demonstrate in a solid state quantum gate based on the electronic excitation of a control atom that it is possible to disentangle the control electron, in the presence of an arbitrary magnetic field and exchange interaction. This will allow multiple gates on a single substrate, providing the basis for a solid state quantum computer.

## 1. Introduction

An electronic spin state can be used to represent one qubit of quantum information, with a general state being the superposition of the spin up and spin down states. In order to perform quantum computation, entanglement between qubits is required. In this paper we consider a solid state implementation proposed by Stoneham *et al* [1] which exploits the presence of defect centres in a solid. The use of defect centres for quantum computation has been experimentally demonstrated by Jelezko *et al* [2] and is particularly attractive since it has the potential for operation at room temperature [1, 2].

The entanglement of the electronic spins states associated with two defects $A$ and $B$ are controlled by the electronic excitation of a control atom $C$. As can be seen in figure 1, when the control electron is in the ground state the wavefunction $W_{CG}$ does not overlap with the wavefunctions $W_{\mathrm{A}}$ and $W_{\mathrm{B}}$. However, when the control electron is put into an excited state the wavefunction $W_{\mathrm{CE}}$ now interacts with both $W_{\mathrm{A}}$ and $W_{\mathrm{B}}$, enabling the spin states of $A$ and $B$ to become entangled. The energy to move between the ground and excited states is provided by a laser pulse of duration $\tau_{\mathrm{pulse}}$ having a frequency resonantly tuned to the electronic excitation energy. The system will continue to evolve in the excited state until after a time $T \gg \tau_{\mathrm{pulse}}$ a second laser pulse is applied to de-excite the control electron, causing it to return to the ground state.

In general the spin states of the de-excited control electron $C$ will remain entangled with the spin states of $A$ and $B$. This results in a gate which is not time invariant, but whose functionality is entangled with all previous qubits and operations involving the gate. While there may be possible advantages to this, from a design perspective it is desirable for the gate

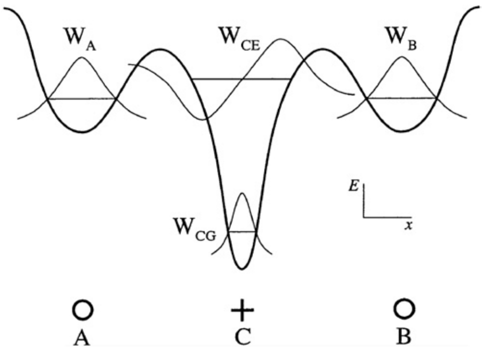


**Figure 1.** Energy diagram for the qubits $A$ and $B$ and control $C$.

to be time invariant. To achieve this we need to find the value of $T$ which minimizes the entanglement between the spin states of the control $C$ and the qubits $A$ and $B$.

## 2. Theoretical model

The interaction between the three electrons may be modelled with an effective Heisenberg interaction [3]. If we choose units such that $\hbar = 1$ then the excited state Hamiltonian is of the form [4]

$$H_{\mathrm{e}} = J_A \boldsymbol{\sigma}_A \cdot \boldsymbol{\sigma}_C + J_B \boldsymbol{\sigma}_B \cdot \boldsymbol{\sigma}_C + B_A \sigma_{Az} + B_B \sigma_{Bz} + B_C \sigma_{Cz} \tag{1}$$

where $\sigma_i$ are the Pauli spin matrices and $\boldsymbol{\sigma} = \sigma_x \mathbf{i} + \sigma_y \mathbf{j} + \sigma_z \mathbf{k}$; $J_A$ is the exchange interaction strength between $A$ and $C$, $J_B$ is the exchange interaction strength between $B$ and $C$, and $B_i = -|\mathbf{B}|\mu_i$, where $\mu_i$ is the magnetic moment of particle $i$, and $\mathbf{B}$ is the magnetic field. If we assume that $A$ and $B$ are identical particles such that $J_A = J_B = J$ say and $B_A = B_B = B$, then the Hamiltonian $H_{\mathrm{e}}$ acting on the state $|\psi\rangle = |CBA\rangle$ is given by

$$H_{\mathrm{e}} = B \cdot I_2 \otimes (I_2 \otimes \sigma_z + \sigma_z \otimes I_2) + B_C \cdot \sigma_z \otimes I_2 \otimes I_2 + J \cdot \sum_{x,y,z} \sigma_i \otimes (I_2 \otimes \sigma_i + \sigma_i \otimes I_2) \tag{2}$$

which is represented by the following $8 \times 8$ matrix:

$$H_{\mathrm{e}} = \begin{bmatrix} 2B + B_c + 2J & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B_c & 0 & 0 & 2J & 0 & 0 & 0 \\ 0 & 0 & B_c & 0 & 2J & 0 & 0 & 0 \\ 0 & 0 & 0 & -2J - 2B + B_c & 0 & 2J & 2J & 0 \\ 0 & 2J & 2J & 0 & -2J + 2B - B_c & 0 & 0 & 0 \\ 0 & 0 & 0 & 2J & 0 & -B_c & 0 & 0 \\ 0 & 0 & 0 & 2J & 0 & 0 & -B_c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2J - 2B - B_c \end{bmatrix}. \tag{3}$$

This matrix may be diagonalized such that $H_{\mathrm{e}} = M^{-1} \Lambda M$, where $\Lambda$ is a diagonal matrix with elements $(3B - fJ + 2J, B - fJ, B - J + J\omega_-, B - J - J\omega_-, -B - J - J\omega_+, -B - J + J\omega_+, -B + fJ, 2J - 3B + fJ)$, with $M$ given by

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & \dfrac{-1+\omega_- + f}{2} & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & \dfrac{-1-\omega_- + f}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{-1-\omega_+ - f}{2} & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & \dfrac{-1+\omega_+ - f}{2} & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

and $f = (B - B_c)/J$, $\omega_+ = \sqrt{f^2 + 2f + 9}$, $\omega_- = \sqrt{f^2 - 2f + 9}$. This diagonal form allows the calculation of $U(t) = \mathrm{e}^{-\mathrm{i}H_c t}$ to be simplified to $U(t) = M^{-1}\mathrm{e}^{-\mathrm{i}\Lambda t}M$, which we write in the form

$$U(t) = \mathrm{e}^{-\mathrm{i}H_c t} = \begin{bmatrix} Q_+(t) & \chi_-(t) \\ \chi_+(t) & Q_-(t) \end{bmatrix} \tag{5}$$

where $Q_+(t)$ and $Q_-(t)$ are the $4 \times 4$ matrices which govern the entanglement between the qubits $A$ and $B$, when the control spin $C$ is initially up or down, respectively. Similarly, $\chi_\pm(t)$ are $4 \times 4$ matrices which govern the entanglement between the qubits and the spin state of the control electron. After an interpulse time $T$ a second laser pulse is applied, causing the control electron to return to the unexcited state such that for $t \geqslant T$, $U(t) = U(T)$. It can be shown that for $t \geqslant T$, the Euclidean norm $||\cdot||$ of the matrices $\chi_+(T)$ and $\chi_-(T)$ are equal and are given by

$$||\chi||^2 = 4\left(\frac{1-\cos(2\tau\omega_-)}{\omega_-^2} + \frac{1-\cos(2\tau\omega_+)}{\omega_+^2}\right) \tag{6}$$

where $\tau = JT$.

## 3. Disentangling the control electron

If we initialize the spin of the control electron to an eigenstate of spin up or spin down, it will be disentangled from the information qubits if we are able to set $||\chi_+(T)|| = 0$ and $||\chi_-(T)|| = 0$. Rodriquez *et al* [4] have shown for a single gate that by carefully choosing the magnetic field $\mathbf{B}$ and the interpulse time $T$ this can be achieved. However, in a quantum computer there may be multiple gates on a single substrate and therefore this approach cannot in general be used, since the magnetic field will be approximately homogeneous, with the only degree of freedom for each gate being the interpulse time $T$. The only exception is when $\mathbf{B} = 0$, where by choosing $JT = n\pi/3$ ($n \in \mathbb{Z}^+$) then $||\chi_+(T)|| = ||\chi_-(T)|| = 0$, with the additional advantage that since $Q_+(T) = Q_-(T)$, the control electron will be disentangled regardless of the initial state. Nevertheless in practice it is desirable to have a magnetic field to create Zeeman splitting between the degenerate energy levels. As such the problem becomes to obtain the value of $T$ which minimizes the Euclidean norm of the matrices $\chi_-(T)$ and $\chi_+(T)$.

In order to minimize the Euclidean norm, given by equation (6), we note that the frequencies $2\omega_-$ and $2\omega_+$ beat with each other, and hence we aim to isolate the beat terms, re-writing $||\chi||^2$ as

$$||\chi||^2 = 4\frac{\omega_+^2 + \omega_-^2}{\omega_+^2\omega_-^2}[1 - \cos(\tau\omega_s)\cos(\tau\omega_d)] - 4\frac{\omega_+^2 - \omega_-^2}{\omega_+^2\omega_-^2}\sin(\tau\omega_s)\sin(\tau\omega_d) \tag{7}$$

where $\omega_s = \omega_+ + \omega_-$ and $\omega_d = \omega_+ - \omega_-$. To minimize $||\chi||$ for an arbitrary value of $f$ we require $\cos(\tau\omega_s)\cos(\tau\omega_d) = 1$ and $\sin(\tau\omega_s)\sin(\tau\omega_d) = 0$, resulting in the conditions $\tau\omega_- = M\pi$ and $\tau\omega_+ = N\pi$, from which we obtain

$$\frac{\omega_-}{\omega_+} = \frac{m}{n} \tag{8}$$

where $m$ and $n$ are integers such that $M = km$ and $N = kn$ and $k$ is a positive integer. The exact value of $k$ will determine the functionality of the gate; however, for the sake of simplicity we consider the first solution with $k = 1$, such that $M$ and $N$ have no common factors.

## 4. Obtaining the optimum interpulse time *T*

Using continued fractions [5] it is possible to obtain rational approximations to $\omega_-/\omega_+$, such that

$$\frac{\omega_-}{\omega_+} \approx \frac{M}{N} \tag{9}$$

which gives two possible values for $\tau$, namely $\tau_- = M\pi/\omega_-$ and $\tau_+ = N\pi/\omega_+$. To find the minimum value of $||\chi||$ we write its derivative in the form

$$\frac{\mathrm{d}||\chi||^2}{\mathrm{d}\tau} = 8\frac{\sin(2[\tau_- + (\tau - \tau_-)]\omega_-)}{\omega_-} + 8\frac{\sin(2[\tau_+ + (\tau - \tau_+)]\omega_+)}{\omega_+} \tag{10}$$

from which, on expanding the first and second terms of the right-hand side about $\tau - \tau_-$ and $\tau - \tau_+$, respectively, and setting the derivative equal to zero, we deduce that the minimum of $||\chi||^2$ occurs at

$$\tau = \frac{\tau_- + \tau_+}{2} = \frac{\pi}{2}\left(\frac{M}{\omega_-} + \frac{N}{\omega_+}\right). \tag{11}$$

The final stage is to estimate the residual norm of $||\chi||$ at the optimum time. By expanding $||\chi||^2$ as a series about $\omega_- = M/N \times \omega_+$, and noting $\omega_-/\omega_+ \approx M/N$, we obtain

$$||\chi|| = 2\pi\left|\frac{M}{\omega_-} - \frac{N}{\omega_+}\right|. \tag{12}$$

## 5. Application to a pair of gates in a homogeneous magnetic field

By way of an example let us consider a pair of gates in a homogeneous magnetic field. Let us suppose that we have two gates, in one of which we have $J_1 = 1$ Grad s$^{-1}$ and in the second $J_2 = 2$ Grad s$^{-1}$. We apply a homogeneous magnetic field to both gates such that $f = 1$ in the first gate (and therefore $f = 0.5$ in the second). What values should $T_1$ and $T_2$ take to minimize the entanglement of the qubits and the spin state of the control electron?

For the first gate $\omega_+ = 2\sqrt{3}$, $\omega_- = 2\sqrt{2}$, and therefore we wish to approximate $\sqrt{2/3}$ as a rational number. Any real number $p$ may be expressed as a continued fraction [5]

$$p = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}} \tag{13}$$

where the $a_n$ are given by the recursive formula

$$a_n = \lfloor r_n \rfloor \tag{14}$$

where $\lfloor x \rfloor$ is the integer part of $x$ rounded towards minus infinity and $r_n$ is given by

$$r_n = \frac{1}{r_{n-1} - a_{n-1}} \tag{15}$$

with the initial conditions $a_0 = \lfloor p \rfloor$ and $r_0 = p$. By taking $n$ terms of the continued fraction expansion of $p$ we can create rational approximations to $p$. Applying this method to $\omega_-/\omega_+ = \sqrt{2/3}$ gives the following continued fraction expansion:

$$\frac{\omega_-}{\omega_+} = \sqrt{\frac{2}{3}} = \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{2 + \cfrac{1}{4 + \cfrac{1}{2 + \cfrac{1}{4 + \cdots}}}}}} \tag{16}$$

which gives rational approximations to $\omega_-/\omega_+$ as

$$\frac{\omega_-}{\omega_+} = \sqrt{\frac{2}{3}} \approx \frac{M}{N} = \left(0, 1, \frac{4}{5}, \frac{9}{11}, \frac{40}{49}, \frac{89}{109}, \frac{396}{485}, \ldots\right). \tag{17}$$

As an example we pick $M = 40$, $N = 49$, which gives $\tau = 44.43$ and therefore $T_1 = \tau/J_1 = 44.4$ ns, with a residual error of $||\chi|| = 0.02$.

Similarly, for the second gate $\omega_+ = \sqrt{41}/2$, $\omega_- = \sqrt{33}/2$, and therefore the continued fraction expansion of $\omega_-/\omega_+$ is given by

$$\frac{\omega_-}{\omega_+} = \sqrt{\frac{33}{41}} = \cfrac{1}{1 + \cfrac{1}{8 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}}} \tag{18}$$

giving rational approximations to $\omega_-/\omega_+$ as

$$\frac{\omega_-}{\omega_+} = \sqrt{\frac{33}{41}} \approx \frac{M}{N} = \left(0, 1, \frac{8}{9}, \frac{9}{10}, \frac{26}{29}, \frac{35}{39}, \frac{61}{68}, \ldots\right). \tag{19}$$

As an example we pick $M = 61$, $N = 68$, which gives $\tau = 66.72$ and therefore $T_2 = \tau/J_2 = 33.4$ ns, with a residual error of $||\chi|| = 0.01$.

## 6. Conclusion

We have shown, in a solid state quantum gate, that it is possible to disentangle the control electron from the information qubits to ensure that the functionality of the gate is time invariant. By choosing an optimum interpulse time $T$ based on the method of continued fractions we have demonstrated that it is always possible to disentangle the control electron from the information qubits. Since both the magnetic field and the exchange interactions are arbitrary, this will allow multiple gates on a single substrate to be interconnected, providing the basis for a solid state quantum computer.

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